

p -adic measures associated with zeta values and p -adic log multiple gamma functions

Tomokazu Kashio*

April 18, 2017

Abstract

We study a relation between two refinements of the rank one abelian Gross-Stark conjecture: For a suitable abelian extension H/F of number fields, a Gross-Stark unit is defined as a p -unit of H satisfying some properties. Let $\tau \in \text{Gal}(H/F)$. Yoshida and the author constructed the symbol $Y_p(\tau)$ by using p -adic log multiple gamma functions, and conjectured that the \log_p of a Gross-Stark unit can be expressed by $Y_p(\tau)$. Dasgupta constructed the symbol $u_T(\tau)$ by using the p -adic multiplicative integration, and conjectured that a Gross-Stark unit can be expressed by $u_T(\tau)$. In this paper, we give an explicit relation between $Y_p(\tau)$ and $u_T(\tau)$.

1 Introduction

Let F be a totally real field, K a CM-field which is abelian over F , S a finite set of places of F . We assume that

- S contains all infinite places of F , all places of F lying above a rational prime p , and all ramified places in K/F .
- Let \mathfrak{p} be the prime ideal corresponding to the p -adic topology on F . (Hence $\mathfrak{p} \in S$.) Then \mathfrak{p} splits completely in K/F .

For $\tau \in \text{Gal}(K/F)$, we consider the partial zeta function

$$\zeta_S(s, \tau) := \sum_{\left(\frac{K/F}{\mathfrak{a}}\right)=\tau, (\mathfrak{a}, S)=1} N\mathfrak{a}^{-s}.$$

Here \mathfrak{a} runs over all integral ideals of F , relatively prime to any finite places in S , whose image under the Artin symbol $\left(\frac{K/F}{*}\right)$ is equal to τ . The series converges for $\text{Re}(s) > 1$, has a meromorphic continuation to the whole s -plane, and is analytic at $s = 0$. Moreover, under our assumption, we see that

2010 *Mathematics subject classification*(s). 11R27, 11R37, 11R42, 11R80, 11S40, 11S80, 33B15.

Key words and phrases. the Gross-Stark conjecture, multiple gamma functions, p -adic measures.

*Tokyo University of Science, kashio_tomokazu@ma.noda.tus.ac.jp

- There exists the p -adic interpolation function $\zeta_{p,S}(s, \tau)$ of $\zeta_S(s, \tau)$.
- $\text{ord}_{s=0}\zeta_S(s, \tau), \text{ord}_{s=0}\zeta_{p,S}(s, \tau) \geq 1$.
- There exist a natural number W , a \mathfrak{p} -unit u of K , which satisfy

$$\log |u^\tau|_{\mathfrak{P}} = -W\zeta'_S(0, \tau) \quad (\tau \in \text{Gal}(K/F)). \quad (1)$$

Here \mathfrak{P} denotes the prime ideal corresponding to the p -adic topology on K , $|x|_{\mathfrak{P}} := N\mathfrak{P}^{-\text{ord}_{\mathfrak{P}}x}$.

Gross conjectured the following p -adic analogue of the rank 1 abelian Stark conjecture:

Conjecture 1 ([Gr, Conjecture 3.13]). *Let u be a \mathfrak{p} -unit characterized by (1) up to roots of unity. Then we have*

$$\log_p N_{K_{\mathfrak{P}}/\mathbb{Q}_p}(u^\tau) = -W\zeta_{p,S}(0, \tau).$$

Dasgupta-Darmon-Pollack [DDP] proved a large part of Conjecture 1. Yoshida and the author, and independently Dasgupta formulated refinements of Conjecture 1: Let \mathfrak{f} be an integral ideal of a totally real field F satisfying $\mathfrak{p} \nmid \mathfrak{f}$, $H_{\mathfrak{f}}$ the narrow ray class field modulo \mathfrak{f} , H the maximal subfield of $H_{\mathfrak{f}}$ where \mathfrak{p} splits completely. Yoshida and the author [KY1] essentially constructed the invariant $Y_p(\tau)$ (Definition 6) for $\tau \in \text{Gal}(H/F)$ by using p -adic log multiple gamma functions. Then [KY1, Conjecture A'] states that $\log_p u^\tau$ (without $N_{K_{\mathfrak{P}}/\mathbb{Q}_p}$) can be expressed by $Y_p(\tau)$. On the other hand, Dasgupta constructed the invariant $u_T(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}})$ (Definition 12-(iv)) by using the multiplicative integration for p -adic measures associated with Shintani's multiple zeta functions. Then [Da, Conjecture 3.21] states that a modified version of u^τ can be expressed by $u_T(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}})$. In [Ka3, Remark 2], the author announced the following relation between these refinements.

Theorem (Theorem 1). *Let η be a “good” prime ideal in the sense of Definition 11. We put $T := \{\eta\}$. Then we have*

$$\log_p(u_\eta(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}})) = -Y_p\left(\left(\frac{H/F}{\mathfrak{b}}\right)\right) + N\eta Y_p\left(\left(\frac{H/F}{\mathfrak{b}\eta^{-1}}\right)\right).$$

In particular, we see that two refinements are consistent (roughly speaking, [Da, Conjecture 3.21] is a further refinement of [KY1, Conjecture A'] by $\ker \log_p$). The aim of this paper is to prove this Theorem.

Let us explain the outline of this paper. In §2, we introduce Shintani's technique of cone decompositions. We obtain a suitable fundamental domain of $F \otimes \mathbb{R}_+/E_{\mathfrak{f},+}$, where $F \otimes \mathbb{R}_+$ denotes the totally positive part of $F \otimes \mathbb{R}$, $E_{\mathfrak{f},+}$ is a subgroup of the group of all totally positive units. We need such fundamental domains in order to construct both of the invariants Y_p , u_T . In §3, we recall the definition and some properties of Y_p , which is essentially defined in [KY1] and slightly modified in [Ka3]. The classical or p -adic log multiple gamma function is defined as the derivative values at $a = 0$ of the classical or p -adic Barnes' multiple zeta function, respectively. Then the invariant $Y_p(\tau, \iota)$ is defined in Definition 6, as a finite sum of the “difference” of p -adic log multiple gamma functions and classical log multiple gamma functions. Conjecture 2 predicts exact values

of $Y_p(\tau, \iota)$. In §4, we also recall some results in [Da]. Dasgupta introduced p -adic measures $\nu_T(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}})$ associated with special values of Shintani's multiple zeta functions, and defined $u_T(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}})$ as the multiplicative integration $\oint_{\mathcal{O}} x d\nu_{\eta}(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, x)$ with certain correction terms. Dasgupta formulated a conjecture (Conjecture 3) on properties of $u_T(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}})$. In §5, we state and prove the main result (Theorem 1) which gives an explicit relation between $Y_p(\tau, \text{id})$ and $\log_p(u_{\eta}(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}))$. Then we will see that Conjectures 2, 3 are consistent in the sense of Corollary 1. The key observation is Lemma 3: Dasgupta's p -adic measure $\nu_{\eta}(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}})$ is originally associated with Shintani's multiple zeta functions. By this Lemma, we can relate $\nu_{\eta}(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}})$ to Barnes' multiple zeta functions and p -adic analogues as in Lemma 4.

2 Shintani domains

Let F be a totally real field of degree n , \mathcal{O}_F the ring of integers of F , \mathfrak{f} an integral ideal of F . We denote by F_+ the set of all totally positive elements in F and put $\mathcal{O}_{F,+} := \mathcal{O}_F \cap F_+$, $E_+ := \mathcal{O}_F^{\times} \cap F_+$. We consider subgroups of E_+ of the following form:

$$E_{\mathfrak{f},+} := \{\epsilon \in E_+ \mid \epsilon \equiv 1 \pmod{\mathfrak{f}}\}.$$

We identify

$$F \otimes \mathbb{R} = \mathbb{R}^n, \quad \sum_{i=1}^k a_i \otimes b_i \mapsto \left(\sum_{i=1}^k \iota(a_i) b_i \right)_{\iota \in \text{Hom}(F, \mathbb{R})},$$

where $\text{Hom}(F, \mathbb{R})$ denotes the set of all real embeddings of F . In particular, the totally positive part

$$F \otimes \mathbb{R}_+ := \mathbb{R}_+^n$$

has a meaning. On the right-hand side, \mathbb{R}_+ denotes the set of all positive real numbers. Let $v_1, \dots, v_r \in \mathcal{O}_F$ be linearly independent. Then we define the cone with basis $\mathbf{v} := (v_1, \dots, v_r)$ as

$$C(\mathbf{v}) := \{\mathbf{t}^t \mathbf{v} \in F \otimes \mathbb{R} \mid \mathbf{t} \in \mathbb{R}_+^r\}.$$

Here we $\mathbf{t}^t \mathbf{v}$ denotes the inner product.

Definition 1. (i) We call a subset $D \subset F \otimes \mathbb{R}_+$ is a Shintani set if it can be expressed as a finite disjoint union of cones:

$$D = \coprod_{i \in J} C(\mathbf{v}_j) \quad (|J| < \infty, \mathbf{v}_j \in \mathcal{O}_{F,+}^{r(j)}, r(j) \in \mathbb{N}).$$

(ii) We consider the natural action $E_{\mathfrak{f},+} \curvearrowright F \otimes \mathbb{R}_+$, $u(a \otimes b) := (ua) \otimes b$. We call a Shintani set D a Shintani domain $\text{mod } E_{\mathfrak{f},+}$ if it is a fundamental domain of $F \otimes \mathbb{R}_+ / E_{\mathfrak{f},+}$:

$$F \otimes \mathbb{R}_+ = \coprod_{\epsilon \in E_{\mathfrak{f},+}} \epsilon D.$$

When $\mathfrak{f} = (1)$, we write $\text{mod } E_+$ instead of $\text{mod } E_{(1),+}$.

Shintani [Sh, Proposition 4] showed that there always exists a Shintani domain.

3 p -adic log multiple gamma functions

We recall the definition and some properties of the symbol Y_p defined in [KY1], [Ka3]. We denote by \mathbb{R}_+ the set of all positive real numbers.

Definition 2. Let $z \in \mathbb{R}_+$, $\mathbf{v} \in \mathbb{R}_+^r$. Barnes' multiple zeta function is defined as

$$\zeta(s, \mathbf{v}, z) := \sum_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^r} (z + \mathbf{m}^t \mathbf{v})^{-s}.$$

This series converges for $\operatorname{Re}(s) > r$, has a meromorphic continuation to the whole s -plane, and is analytic at $s = 0$. Then Barnes' multiple gamma function is defined as

$$\Gamma(z, \mathbf{v}) := \exp \left(\frac{\partial}{\partial s} \zeta(s, \mathbf{v}, z) \Big|_{s=0} \right).$$

Note that this definition is modified from that given by Barnes. For the proof and details, see [Yo, Chap I, §1]. Throughout this paper, we regard each number field as a subfield of $\overline{\mathbb{Q}}$, and fix two embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Here \mathbb{C}_p denotes the p -adic completion of the algebraic closure of \mathbb{Q}_p . We denote by $\mu_{(p)}$ the group of all roots of unity of prime-to- p order. Let $\operatorname{ord}_p: \mathbb{C}_p^\times \rightarrow \mathbb{Q}$, $\theta_p: \mathbb{C}_p^\times \rightarrow \mu_{(p)}$ be unique group homomorphisms satisfying

$$|p^{-\operatorname{ord}_p(z)} \theta_p(z)^{-1} z|_p < 1 \quad (z \in \mathbb{C}_p^\times). \quad (2)$$

Definition 3. Let $z \in \overline{\mathbb{Q}}$, $\mathbf{v} \in (\overline{\mathbb{Q}}^\times)^r$. We assume that

$$\begin{aligned} z \in \mathbb{R}_+, \mathbf{v} \in \mathbb{R}_+^r \text{ via the embedding } \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \\ \operatorname{ord}_p(z) < \operatorname{ord}_p(v_1), \dots, \operatorname{ord}_p(v_r) \text{ via the embedding } \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p. \end{aligned} \quad (3)$$

Then we denote by $\zeta_p(s, \mathbf{v}, z)$ ($s \in \mathbb{Z}_p - \{1, 2, \dots, r\}$) the p -adic multiple zeta function characterized by

$$\zeta_p(-m, \mathbf{v}, z) = p^{-\operatorname{ord}_p(z)m} \theta_p(z)^{-m} \zeta(-m, \mathbf{v}, z) \quad (m \in \mathbb{Z}_{\geq 0}). \quad (4)$$

We define the p -adic log multiple gamma function as

$$L\Gamma_p(z, \mathbf{v}) := \frac{\partial}{\partial s} \zeta_p(s, \mathbf{v}, z) \Big|_{s=0}.$$

The construction of $\zeta_p(s, \mathbf{v}, z)$ is due to Cassou-Noguès [CN1]. The author defined and studied $L\Gamma_p(z, \mathbf{v})$ in [Ka1]. See [Ka3, §2] for a short survey.

Definition 4. Let F be a totally real field, \mathfrak{f} an integral ideal, $D = \coprod_{j \in J} C(\mathbf{v}_j)$ ($\mathbf{v}_j \in \mathcal{O}_{F,+}^{r(j)}$) a Shintani domain $\bmod E_+$. We denote by $\operatorname{Hom}(F, \mathbb{R})$ (resp. $\operatorname{Hom}(F, \mathbb{C}_p)$) the set of all embeddings of F into \mathbb{R} (resp. \mathbb{C}_p). Since we fixed embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, we may identify

$$\operatorname{Hom}(F, \mathbb{R}) = \operatorname{Hom}(F, \mathbb{C}_p).$$

- (i) We denote by $C_{\mathfrak{f}}$ the narrow ideal class group modulo \mathfrak{f} , by $H_{\mathfrak{f}}$ the narrow ray class field modulo \mathfrak{f} . In particular, the Artin map induces

$$C_{\mathfrak{f}} \cong \text{Gal}(H_{\mathfrak{f}}/F).$$

- (ii) Let $\pi: C_{\mathfrak{f}} \rightarrow C_{(1)}$ be the natural projection. For each $c \in C_{\mathfrak{f}}$, we take an integral ideal \mathfrak{a}_c satisfying

$$\mathfrak{a}_c \mathfrak{f} \in \pi(c).$$

- (iii) For $c \in C_{\mathfrak{f}}$, $\mathbf{v} \in \mathcal{O}_F^r$, we put

$$R(c, \mathbf{v}) := R(c, \mathbf{v}, \mathfrak{a}_c) := \{\mathbf{x} \in (\mathbb{Q} \cap (0, 1])^r \mid \mathcal{O}_F \supset (\mathbf{x}^t \mathbf{v}) \mathfrak{a}_c \mathfrak{f} \in c\}.$$

- (iv) For $c \in C_{\mathfrak{f}}$, $\iota \in \text{Hom}(F, \mathbb{R})$, we define

$$G(c, \iota) := G(c, \iota, D, \mathfrak{a}_c) := \sum_{j \in J} \sum_{\mathbf{x} \in R(c, \mathbf{v}_j)} \log \Gamma(\iota(\mathbf{x}^t \mathbf{v}_j), \iota(\mathbf{v}_j)).$$

- (v) For $\iota \in \text{Hom}(F, \mathbb{R})$ ($= \text{Hom}(F, \mathbb{C}_p)$), we put

$$\mathfrak{p}_{\iota} := \{z \in \mathcal{O}_F \mid |\iota(z)|_p < 1\}.$$

Note that the prime ideal $\iota(\mathfrak{p}_{\iota})$ corresponds to the p -adic topology on $\iota(F) \subset \mathbb{C}_p$.

- (vi) Assume that $\mathfrak{p}_{\iota} \mid \mathfrak{f}$. For $c \in C_{\mathfrak{f}}$, $\iota \in \text{Hom}(F, \mathbb{R})$, we define

$$G_p(c, \iota) := G_p(c, \iota, D, \mathfrak{a}_c) := \sum_{j \in J} \sum_{\mathbf{x} \in R(c, \mathbf{v}_j)} L\Gamma_p(\iota(\mathbf{x}^t \mathbf{v}_j), \iota(\mathbf{v}_j)).$$

Note that $(\iota(\mathbf{x}^t \mathbf{v}_j), \iota(\mathbf{v}_j))$ satisfies the assumption (3) whenever $\mathfrak{p}_{\iota} \mid \mathfrak{f}$, $\mathbf{x} \in R(c, \mathbf{v}_j)$.

The following map $[\]_p$ is well-defined by [KY1, Lemma 5.1].

Definition 5. We denote by $\overline{\mathbb{Q}} \log_p \overline{\mathbb{Q}}^{\times}$ (resp. $\overline{\mathbb{Q}} \log \overline{\mathbb{Q}}^{\times}$) the $\overline{\mathbb{Q}}$ -subspace of \mathbb{C}_p (resp. \mathbb{C}) generated by $\log_p b$ (resp. $\pi, \log b$) with $b \in \overline{\mathbb{Q}}^{\times}$. We define a $\overline{\mathbb{Q}}$ -linear map $[\]_p$ by

$$[\]_p: \overline{\mathbb{Q}} \log \overline{\mathbb{Q}}^{\times} \rightarrow \overline{\mathbb{Q}} \log_p \overline{\mathbb{Q}}^{\times}, \quad a \log b \mapsto a \log_p b, \quad a\pi \mapsto 0 \quad (a, b \in \overline{\mathbb{Q}}, b \neq 0).$$

Lemma 1. Let H be an intermediate field of $H_{\mathfrak{f}}/F$, \mathfrak{q} a prime ideal of F , relatively prime to \mathfrak{f} , splitting completely in H/F . Then we have

$$\sum_{c \in C_{\mathfrak{f}\mathfrak{q}}, \text{Art}(\bar{c})|_H = \tau} G(c, \iota) \in \overline{\mathbb{Q}} \log \overline{\mathbb{Q}}^{\times} \quad (\tau \in \text{Gal}(H/F)).$$

Here c runs over all ideal classes whose images under the composite map $C_{\mathfrak{f}\mathfrak{q}} \rightarrow C_{\mathfrak{f}} \rightarrow \text{Gal}(H_{\mathfrak{f}}/F) \rightarrow \text{Gal}(H/F)$ is equal to τ .

Proof. We put $W(c, \iota) := W(\iota(c))$ in [KY1, (4.3)], $V(c, \iota) := V(\iota(c))$ in [KY1, (1.6)], and $X(c, \iota) := G(c, \iota) + W(c, \iota) + V(c, \iota)$. Here we consider the ideal class group $C_{\iota(\mathfrak{f})}$ of $\iota(F)$ modulo $\iota(\mathfrak{f})$. Then $\iota(c)$ denotes the image of $c \in C_{\mathfrak{f}}$ in $C_{\iota(\mathfrak{f})}$ under the natural map. By the definition [KY1, (4.3)] and [KY2, Appendix I, Theorem], we have $W(c, \iota), V(c, \iota) \in \overline{\mathbb{Q}} \log \overline{\mathbb{Q}}^\times$. Moreover [KY1, Lemma 5.5] states that

$$\sum_{c \in C_{\mathfrak{f}\mathfrak{q}}} \chi_{\mathfrak{q}}(c) X(c, \iota) \in \overline{\mathbb{Q}} \log \overline{\mathbb{Q}}^\times \quad (\chi \in \hat{C}_{\mathfrak{f}}, \chi([\mathfrak{q}]) = 1).$$

Here $\chi_{\mathfrak{q}}, [\mathfrak{q}]$ denote the composite map $C_{\mathfrak{f}\mathfrak{q}} \rightarrow C_{\mathfrak{f}} \xrightarrow{\chi} \mathbb{C}^\times$, the ideal class $\in C_{\mathfrak{f}}$ of \mathfrak{q} , respectively. Therefore, when H is the fixed subfield under $(\frac{H_{\mathfrak{f}}/F}{\mathfrak{q}})$, it follows from the orthogonality of characters. The general case follows from this case immediately. \square

Definition 6. Let H be an intermediate field of $H_{\mathfrak{f}}/F$. Assume that $\mathfrak{p}_\iota \nmid \mathfrak{f}$ and that \mathfrak{p}_ι splits completely in H/F . Then we define

$$Y_p(\tau, \iota) := \sum_{c \in C_{\mathfrak{f}\mathfrak{p}_\iota}, \text{Art}(\bar{c})|_H = \tau} G_p(c, \iota) - \left[\sum_{c \in C_{\mathfrak{f}\mathfrak{p}_\iota}, \text{Art}(\bar{c})|_H = \tau} G(c, \iota) \right]_p \quad (\tau \in \text{Gal}(H/F)).$$

When $\iota = \text{id}$, we drop the symbol ι : $Y_p(\tau) := Y_p(\tau, \text{id})$.

By [KY1, Proposition 5.6] (and the orthogonality of characters), we see that $Y_p(\tau, \iota)$ depends only on $H, \mathfrak{f}, \tau, \iota$, not on D, \mathfrak{a}_c 's. We formulated a conjecture [KY1, Conjecture A'], which is equivalent to the following Conjecture 2 by [Ka3, Proposition 6-(ii)].

Conjecture 2. Let $H_{\mathfrak{f}}/H/F$ be as above: we assume that

\mathfrak{p}_ι does not divide \mathfrak{f} , splits completely in H/F .

We take a lift $\tilde{\iota}: H \rightarrow \mathbb{C}_p$ of $\iota: F \rightarrow \mathbb{C}_p$ and put $\mathfrak{p}_{H, \tilde{\iota}} := \{z \in \mathcal{O}_H \mid |\tilde{\iota}(z)|_p < 1\}$. Let $\alpha_{H, \tilde{\iota}}$ be a generator of the principal ideal $\mathfrak{p}_{H, \tilde{\iota}}^{h_H}$, where h_H denotes the class number. Then we have

$$Y_p(\tau, \iota) = \frac{-1}{h_H} \sum_{c \in C_{\mathfrak{f}}} \zeta(0, c^{-1}) \log_p \tilde{\iota} \left(\alpha_{H, \tilde{\iota}}^{\tau \text{Art}(c)} \right).$$

Remark 1. Roughly speaking, the above conjecture states a relation between the ratios [p -adic multiple gamma functions : multiple gamma functions] and Stark units associated with the finite place \mathfrak{p}_ι . We also studied a relation between the same ratios and Stark units associated with real places in [Ka3]. We found a more significant relation between the ratios [p -adic gamma function : gamma function] and cyclotomic units in [Ka2].

We rewrite the definition of Y_p for later use.

Definition 7. Let R be a subset of F_+ . We assume that R can be expressed in the following form:

$$R = \coprod_{i=1}^k \{(\mathbf{x}_i + \mathbf{m})^t \mathbf{v}_i \mid \mathbf{m} \in \mathbb{Z}_{\geq 0}^{r_i}\} \quad (\mathbf{x}_i \in \mathbb{Q}_+^{r_i}, \mathbf{v}_i \in F_+^{r_i}).$$

(i) We define

$$\zeta_\iota(s, R) := \sum_{z \in R} \iota(z)^{-s} := \sum_{i=1}^k \zeta(s, \iota(\mathbf{v}_i), \iota(\mathbf{x}_i {}^t \mathbf{v}_i)),$$

$$L\Gamma_\iota(R) := \frac{\partial}{\partial s} \zeta_\iota(s, R)|_{s=0} = \sum_{i=1}^k \log \Gamma(\iota(\mathbf{x}_i {}^t \mathbf{v}_i), \iota(\mathbf{v}_i)).$$

(ii) Additionally we assume that each $(\iota(\mathbf{x}_i {}^t \mathbf{v}_i), \iota(\mathbf{v}_i))$ satisfies (3). Then there exists the p -adic interpolation function

$$\zeta_{\iota,p}(s, R) := \sum_{i=1}^k \zeta_p(s, \iota(\mathbf{v}_i), \iota(\mathbf{x}_i {}^t \mathbf{v}_i))$$

of $\zeta_\iota(s, R)$. We define

$$L\Gamma_{\iota,p}(R) := \frac{\partial}{\partial s} \zeta_{\iota,p}(s, R)|_{s=0} = \sum_{i=1}^k L\Gamma_p(\iota(\mathbf{x}_i {}^t \mathbf{v}_i), \iota(\mathbf{v}_i)).$$

When $\iota = \text{id}$, we drop the symbol ι .

It follows that, for any Shintani domain $D \bmod E_+$ and for any integral ideals \mathfrak{a}_c satisfying $\mathfrak{a}_c \mathfrak{f} \in \pi(c)$, we have

$$Y_p(\tau, \iota) = \sum_{c \in C_{\mathfrak{fp}_\iota}, \text{Art}(\bar{c})|_H = \tau} L\Gamma_{\iota,p}(R_c) - \left[\sum_{c \in C_{\mathfrak{fp}_\iota}, \text{Art}(\bar{c})|_H = \tau} L\Gamma_\iota(R_c) \right]_p, \quad (5)$$

where we put $R_c := \{z \in D \mid \mathcal{O}_F \supset z\mathfrak{a}_c \mathfrak{fp}_\iota \in c\}$. We will use the following properties of the classical or p -adic multiple gamma functions in the proof of Theorem 1.

Proposition 1. (i) Let R be as in Definition 7-(i), $\alpha \in F_+$. Then we have

$$L\Gamma_\iota(R) - L\Gamma_\iota(\alpha R) = \zeta_\iota(0, R) \log \iota(\alpha).$$

(ii) Let R be as in Definition 7-(ii), $\alpha \in F_+$. Then we have

$$L\Gamma_{\iota,p}(R) - L\Gamma_{\iota,p}(\alpha R) = \zeta_\iota(0, R) \log_p \iota(\alpha).$$

Proof. The assertions follow from $\zeta_\iota(s, \alpha R) = \iota(\alpha)^{-s} \zeta_\iota(s, R)$ immediately. \square

We also recall Shintani's multiple zeta functions in [Sh, (1.1)] which we need in subsequent sections.

Definition 8. (i) Let $A = (a_{ij})$ be an $(l \times r)$ -matrix with $a_{ij} \in \mathbb{R}_+$, $\mathbf{x} \in \mathbb{R}_+^r$, $\boldsymbol{\chi} = (\chi_1, \dots, \chi_r) \in (\mathbb{C}^\times)^r$ with $|\chi_i| \leq 1$. Then Shintani's multiple zeta function is defined as

$$\zeta(s, A, \mathbf{x}, \boldsymbol{\chi}) := \sum_{(m_1, \dots, m_r) \in \mathbb{Z}_{\geq 0}^r} \left(\prod_{j=1}^r \chi_j^{m_j} \right) \left(\prod_{i=1}^l \left(\sum_{j=1}^r a_{ij}(m_j + x_j) \right) \right)^{-s}.$$

This series converges for $\text{Re}(s) > \frac{r}{l}$, has a meromorphic continuation to the whole s -plane, is analytic at $s = 0$.

(ii) Let $\mathbf{x}, \boldsymbol{\chi}$ be as in (i). For $\mathbf{v} = (v_1, \dots, v_r) \in F_+^r$, we consider two kinds of Shintani's multiple zeta functions:

(a) Shintani's multiple zeta function with $l = 1$:

$$\zeta(s, \mathbf{v}, \mathbf{x}, \boldsymbol{\chi}) = \sum_{(m_1, \dots, m_r) \in \mathbb{Z}_{\geq 0}^r} \left(\prod_{j=1}^r \chi_j^{m_j} \right) \left(\sum_{j=1}^r v_j(m_j + x_j) \right)^{-s}.$$

Here we consider $v_i \in F_+ \xrightarrow{\text{id}} \mathbb{R}_+$.

(b) Let A be the $(n \times r)$ -matrix whose row vectors are $\iota(\mathbf{v}_i)$ ($\iota \in \text{Hom}(F, \mathbb{R})$, $n := [F : \mathbb{Q}]$). Then we put

$$\zeta_N(s, \mathbf{v}, \mathbf{x}, \boldsymbol{\chi}) := \zeta(s, A, \mathbf{x}, \boldsymbol{\chi}) = \sum_{(m_1, \dots, m_r) \in \mathbb{Z}_{\geq 0}^r} \left(\prod_{j=1}^r \chi_j^{m_j} \right) N \left(\sum_{j=1}^r v_j(m_j + x_j) \right)^{-s}.$$

(iii) Let R be as in Definition 7-(i). We define

$$\zeta_N(s, R) := \sum_{z \in R} N z^{-s} := \sum_{i=1}^k \zeta_N(s, \mathbf{v}_i, \mathbf{x}_i, (1, \dots, 1)).$$

4 p -adic measures associated with zeta values

We consider the following two kinds of integration.

Definition 9. Let K be a finite extension of \mathbb{Q}_p , O the ring of integers of K , P the maximal ideal of O .

(i) We say ν is a p -adic measure on O if for each open compact subset $U \subset O$, it takes the value $\nu(U) \in K$ satisfying

(a) $\nu(U \amalg U') = \nu(U) + \nu(U')$ for disjoint open compact subsets U, U' .

(b) $|\nu(U)|_p$'s are bounded.

We say a p -adic measure ν is a \mathbb{Z} -valued measure if $\nu(U) \in \mathbb{Z}$.

(ii) Let ν be a p -adic measure, $f: O \rightarrow O$ a continuous map. We define

$$\int_O f(x) d\nu(x) := \lim_{\leftarrow} \sum_{\bar{a} \in O/P^m} \nu(f^{-1}(a + P^m)) f(a) \in \lim_{\leftarrow} O/P^m = O.$$

(iii) Let ν be a \mathbb{Z} -valued measure, $f: (O - P^e) \rightarrow (O - P^e)$ a continuous map ($e \in \mathbb{N}$). We define

$$\oint_{O-P^e} f(x) d\nu(x) := \lim_{\leftarrow} \prod_{\bar{a} \in (O-P^e)/(1+P^m)} f(a)^{\nu(f^{-1}(a+P^m))} \in O.$$

We recall the setting in [Da]. Let F be a totally real field of degree n , \mathfrak{f} an integral ideal of F , $\mathfrak{p} := \mathfrak{p}_{\text{id}}$ the prime ideal corresponding to the p -adic topology on F induced by $\text{id}: F \hookrightarrow \mathbb{C}_p$. We assume that $\mathfrak{p} \nmid \mathfrak{f}$.

Definition 10 ([Da, Definitions 3.8, 3.9]). *Let η be a prime ideal of F .*

- (i) *We say η is good for a cone $C(v_1, \dots, v_r)$ if $v_i \in \mathcal{O}_F - \eta$ and if $N\eta$ is a rational prime (i.e., the residue degree = 1).*
- (ii) *We say η is good for a Shintani set D if it can be expressed as a finite disjoint union of cones for which η is good.*

Definition 11 ([Da, Definitions 3.13, 3.16, Conjecture 3.21]). *We take an element $\pi \in \mathcal{O}_{F,+}$, a prime ideal η , a Shintani domain $\mathcal{D}_{\mathfrak{f}} \bmod E_{\mathfrak{f},+}$ satisfying the following conditions.*

- (i) *Let e be the order of \mathfrak{p} in $C_{\mathfrak{f}}$. We fix a generator $\pi \in \mathfrak{p}^e$ satisfying $\pi \in \mathcal{O}_{F,+}$, $\pi \equiv 1 \bmod \mathfrak{f}$.*
- (ii) *$N\eta \geq n + 2$ and $(N\eta, \mathfrak{f}\mathfrak{p}) = 1$.*
- (iii) *The residue degree of $\eta = 1$ and the ramification degree of $\eta \leq N\eta - 2$.*
- (iv) *η is “simultaneously” good for $\mathcal{D}_{\mathfrak{f}}, \pi^{-1}\mathcal{D}_{\mathfrak{f}}$ in the following sense: There exist vectors $\mathbf{v}_j \in (\mathcal{O}_{F,+} - \eta)^{r(j)}$, units $\epsilon_j \in E_{\mathfrak{f},+}$ ($j \in J'$, $|J'| < \infty$) satisfying*

$$\mathcal{D}_{\mathfrak{f}} = \coprod_{j \in J'} C(\mathbf{v}_j), \quad \pi^{-1}\mathcal{D}_{\mathfrak{f}} = \coprod_{j \in J'} \epsilon_j C(\mathbf{v}_j).$$

Remark 2. Dasgupta [Da] took a suitable set T of prime ideals instead of one prime ideal η . In this article, we assume that $|T| = 1$ for simplicity.

We denote by $F_{\mathfrak{p}}, \mathcal{O}_{F_{\mathfrak{p}}}$ the completion of F at \mathfrak{p} , the ring of integers of $F_{\mathfrak{p}}$ respectively.

Definition 12 ([Da, Definitions 3.13, 3.17]). *Let $\pi, \eta, \mathcal{D}_{\mathfrak{f}}$ be as in Definition 11, \mathfrak{b} a fractional ideal of F relatively prime to $\mathfrak{f}\mathfrak{p}N\eta$. We put*

$$F_{\mathfrak{f}}^{\times} := \{z \in F^{\times} \mid z \equiv 1 \bmod^* \mathfrak{f}\}.$$

- (i) *For an open compact subset $\mathcal{U} \subset \mathcal{O}_{F_{\mathfrak{p}}}$, a Shintani set D , we put*

$$\begin{aligned} \nu(\mathfrak{b}, D, \mathcal{U}) &:= \zeta_N(0, F_{\mathfrak{f}}^{\times} \cap \mathfrak{b}^{-1} \cap D \cap \mathcal{U}), \\ \nu_{\eta}(\mathfrak{b}, D, \mathcal{U}) &:= \nu(\mathfrak{b}, D, \mathcal{U}) - N\eta \nu(\mathfrak{b}\eta^{-1}, \mathcal{D}, \mathcal{U}). \end{aligned}$$

Here $\zeta_N(s, R)$ is defined in Definition 8. By [Da, Proposition 3.12] we see that

- *When η is good for D , we have $\nu_{\eta}(\mathfrak{b}, D, \mathcal{U}) \in \mathbb{Z}[N\eta^{-1}]$.*
- *When η is good for D and $N\eta \geq n + 2$, we have $\nu_{\eta}(\mathfrak{b}, D, \mathcal{U}) \in \mathbb{Z}$.*

- (ii) Assume that η is good for D and that $\eta \nmid p$. We define a p -adic measure $\nu_\eta(\mathfrak{b}, D)$ on \mathcal{O}_{F_p} by

$$\nu_\eta(\mathfrak{b}, D)(\mathcal{U}) := \nu_\eta(\mathfrak{b}, D, \mathcal{U}).$$

Under the assumption of Definition 11, $\nu_\eta(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}})$ is a \mathbb{Z} -valued measure.

- (iii) For $\tau \in \text{Gal}(H_{\mathfrak{f}}/F)$, we put

$$\begin{aligned} \zeta_{\mathfrak{f}}(s, \tau) &:= \sum_{\mathfrak{a} \subset \mathcal{O}_F, \left(\frac{H_{\mathfrak{f}}/F}{\mathfrak{a}}\right)=\tau, (\mathfrak{a}, \mathfrak{f})=1} N\mathfrak{a}^{-s}, \\ \zeta_{\mathfrak{f}, \eta}(s, \tau) &:= \zeta_{\mathfrak{f}}(s, \tau) - N\eta^{1-s} \zeta_{\mathfrak{f}}\left(s, \tau\left(\frac{H_{\mathfrak{f}}/F}{\eta^{-1}}\right)\right). \end{aligned}$$

Here $H_{\mathfrak{f}}$ denotes the narrow ray class field modulo \mathfrak{f} .

- (iv) We define

$$\begin{aligned} \epsilon_\eta(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \pi) &:= \prod_{\epsilon \in E_{\mathfrak{f},+}} \epsilon^{\nu_\eta(\mathfrak{b}, \epsilon \mathcal{D}_{\mathfrak{f}} \cap \pi^{-1} \mathcal{D}_{\mathfrak{f}}, \mathcal{O}_{F_p})} \in E_{\mathfrak{f},+}, \\ u_\eta(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}) &:= \epsilon_\eta(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \pi) \pi^{\zeta_{\mathfrak{f}, \eta}(0, \left(\frac{H_{\mathfrak{f}}/F}{\mathfrak{b}}\right))} \int_{\mathbf{O}} x \, d\nu_\eta(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, x) \in F_p^\times, \end{aligned}$$

where $\mathbf{O} := \mathcal{O}_{F_p} - \pi \mathcal{O}_{F_p}$. The product in the first line is actually a finite product since $\nu_\eta(\mathfrak{b}, \epsilon \mathcal{D}_{\mathfrak{f}} \cap \pi^{-1} \mathcal{D}_{\mathfrak{f}}, \mathcal{O}_{F_p}) = 0$ for all but finite $\epsilon \in E_{\mathfrak{f},+}$.

Conjecture 3 ([Da, Conjecture 3.21]). Let $\pi, \eta, \mathcal{D}_{\mathfrak{f}}$ be as in Definition 11, H the fixed subfield of $H_{\mathfrak{f}}$ under $\left(\frac{H_{\mathfrak{f}}/F}{\mathfrak{p}}\right)$.

- (i) Let $\tau \in \text{Gal}(H/F)$. For a fractional ideal \mathfrak{b} relatively prime to $\mathfrak{f}\mathfrak{p}N\eta$ satisfying $\left(\frac{H/F}{\mathfrak{b}}\right) = \tau$, we put

$$u_\eta(\tau) := u_\eta(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}).$$

Then $u_\eta(\tau)$ depends only on \mathfrak{f}, τ, η , not on the choices of $\mathcal{D}_{\mathfrak{f}}, \mathfrak{b}$.

- (ii) For any $\tau \in \text{Gal}(H/F)$, $u_\eta(\tau)$ is a \mathfrak{p} -unit of H satisfying $u_\eta(\tau) \equiv 1 \pmod{\eta}$.
(iii) For any $\tau, \tau' \in \text{Gal}(H/F)$, we have $u_\eta(\tau\tau') = u_\eta(\tau)^{\tau'}$.

5 The main results

We keep the notation in the previous sections: Let F be a totally real field of degree n , $H_{\mathfrak{f}}$ the narrow ray class field modulo \mathfrak{f} . We assume that the prime ideal \mathfrak{p} corresponding to the p -adic topology on F does not divide \mathfrak{f} . Let H be the fixed subfield of $H_{\mathfrak{f}}$ under $\left(\frac{H_{\mathfrak{f}}/F}{\mathfrak{p}}\right)$. For $\tau \in \text{Gal}(H/F)$, let $Y_p(\tau) := Y_p(\tau, \text{id})$ be as in Definition 6. For a fractional ideal \mathfrak{b} relatively prime to $\mathfrak{f}\mathfrak{p}N\eta$, let $u_\eta(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}})$ be as in Definition 12-(iv).

Theorem 1. *We have*

$$\log_p(u_\eta(\mathfrak{b}, \mathcal{D}_\mathfrak{f})) = -Y_p\left(\left(\frac{H/F}{\mathfrak{b}}\right)\right) + N\eta Y_p\left(\left(\frac{H/F}{\mathfrak{b}\eta^{-1}}\right)\right).$$

Corollary 1. *Conjecture 2 implies*

$$u_\eta(\mathfrak{b}, \mathcal{D}_\mathfrak{f})^{h_H} \equiv \prod_{\sigma \in \text{Gal}(H/F)} \alpha_H^{\zeta_{\mathfrak{f}, \eta}(0, \sigma^{-1})\left(\frac{H/F}{\mathfrak{b}}\right)\sigma} \pmod{\ker \log_p},$$

where \mathfrak{p}_H , h_H , α_H are the prime ideal of H corresponding to the p -adic topology on H , the class number of H , a generator of $\mathfrak{p}_H^{h_H}$.

We prepare some Lemmas in order prove this Theorem.

Lemma 2 ([CN2, Théorème 13]). *Let $\mathbf{v} \in F_+^r$, $z \in F_+$, $\boldsymbol{\xi} = (\xi_1, \dots, \xi_r)$ with ξ_i roots of unity, $\neq 1$. For $k \in \mathbb{Z}_{\geq 0}$, we have*

$$\begin{aligned} \zeta_N(-k, \mathbf{v}, z, \boldsymbol{\xi}) &= \sum_{\mathbf{m}=(m_1, \dots, m_r) \in \mathbb{N}^r} \frac{\sum_{\mathbf{l}=(l_1, \dots, l_r), 1 \leq l_i \leq m_i} \left\{ \begin{matrix} \mathbf{m} \\ \mathbf{l} \end{matrix} \right\} N(z - \mathbf{l}^t \mathbf{v})^k}{(1 - \boldsymbol{\xi})^{\mathbf{m}}}, \\ \zeta(-k, \mathbf{v}, z, \boldsymbol{\xi}) &= \sum_{\mathbf{m}=(m_1, \dots, m_r) \in \mathbb{N}^r} \frac{\sum_{\mathbf{l}=(l_1, \dots, l_r), 1 \leq l_i \leq m_i} \left\{ \begin{matrix} \mathbf{m} \\ \mathbf{l} \end{matrix} \right\} (z - \mathbf{l}^t \mathbf{v})^k}{(1 - \boldsymbol{\xi})^{\mathbf{m}}}. \end{aligned}$$

Here we put $(1 - \boldsymbol{\xi})^{\mathbf{m}} := \prod_{i=1}^r (1 - \xi_i)^{m_i}$, $\left\{ \begin{matrix} \mathbf{m} \\ \mathbf{l} \end{matrix} \right\} := \prod_{i=1}^r \left((-1)^{l_i-1} \binom{m_i-1}{l_i-1} \right)$ with the binomial coefficient $\binom{m_i-1}{l_i-1}$. The sum over \mathbf{m} is actually a finite sum since we have $\sum_{\mathbf{l}} \left\{ \begin{matrix} \mathbf{m} \\ \mathbf{l} \end{matrix} \right\} N(z - \mathbf{l}^t \mathbf{v})^k = \sum_{\mathbf{l}} \left\{ \begin{matrix} \mathbf{m} \\ \mathbf{l} \end{matrix} \right\} (z - \mathbf{l}^t \mathbf{v})^k = 0$ if m_i is large enough.

Lemma 3. *Let $\nu_\eta(\mathfrak{b}, D, \mathcal{U})$ be as in Definition 12-(i). Assume that η is good for D . Then we have*

$$\nu_\eta(\mathfrak{b}, D, \mathcal{U}) = \zeta(0, F_\mathfrak{f}^\times \cap \mathfrak{b}^{-1} \cap D \cap \mathcal{U}) - N\eta \zeta(0, F_\mathfrak{f}^\times \cap \mathfrak{b}^{-1} \eta \cap D \cap \mathcal{U}).$$

Proof. It is enough to show the statement when

- D is a cone $C(\mathbf{v})$ with $\mathbf{v} = (v_1, \dots, v_r)$, $v_i \in \mathcal{O}_F - \eta$.
- \mathcal{U} is of the form $a + \mathfrak{p}^m \mathcal{O}_{F_\mathfrak{p}}$ ($m \in \mathbb{N}$, $a \in \mathcal{O}_{F_\mathfrak{p}}$).

Put $R := F_\mathfrak{f}^\times \cap \mathfrak{b}^{-1} \cap C(\mathbf{v}) \cap (a + \mathfrak{p}^m \mathcal{O}_{F_\mathfrak{p}})$. By definition we have

$$\begin{aligned} \nu_\eta(\mathfrak{b}, C(\mathbf{v}), a + \mathfrak{p}^m \mathcal{O}_{F_\mathfrak{p}}) &= \zeta_N(0, R) - N\eta \zeta_N(0, \{z \in R \mid \text{ord}_\eta z > 0\}) \\ &= \left[\sum_{z \in R} N z^{-s} - N\eta \sum_{z \in R, \text{ord}_\eta z > 0} N z^{-s} \right]_{s=0}. \end{aligned} \tag{6}$$

Let L be a positive integer satisfying $L \in \mathfrak{fp}^m \mathfrak{b}^{-1}$, $(\eta, L) = 1$. Then we have

$$\sum_{z \in R} Nz^{-s} = \sum_{\mathbf{x} \in R_a} \sum_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^r} N((\mathbf{x} + \mathbf{m})^t(L\mathbf{v}))^{-s},$$

$$R_a := \{\mathbf{x} \in (\mathbb{Q} \cap (0, 1])^r \mid \mathbf{x}^t(L\mathbf{v}) \in F_{\mathfrak{f}}^{\times} \cap \mathfrak{b}^{-1} \cap (a + \mathfrak{p}^m \mathcal{O}_{F_{\mathfrak{p}}})\}.$$

Since $N\eta$ is a rational prime, the following homomorphism is a surjection.

$$\mathbb{Z} \rightarrow \mathbb{Z}/N\eta \cong \mathcal{O}_F/\eta \cong \mathcal{O}_{F(\eta)}/\eta \mathcal{O}_{F(\eta)}.$$

Here we denote the localization of \mathcal{O}_F at η by $\mathcal{O}_{F(\eta)}$. Hence for each $\mathbf{x} \in R_a$, there exists an integer $n_{\mathbf{x}}$ satisfying $\mathbf{x}^t \mathbf{v} \equiv n_{\mathbf{x}} \pmod{\eta \mathcal{O}_{F(\eta)}}$. Similarly we take n_i satisfying $Lv_i \equiv n_i \pmod{\eta \mathcal{O}_{F(\eta)}}$ and put $\mathbf{n}_{L\mathbf{v}} := (n_1, \dots, n_r)$. Then the following are equivalent:

$$\text{ord}_{\eta}((\mathbf{x} + \mathbf{m})^t(L\mathbf{v})) > 0 \Leftrightarrow n_{\mathbf{x}} + \mathbf{m}^t \mathbf{n}_{L\mathbf{v}} \equiv 0 \pmod{N\eta}.$$

Let ζ be a primitive $N\eta$ th root of unity. We put $\xi_{\mathbf{x}} := \zeta^{n_{\mathbf{x}}}$, $\xi_i := \zeta^{n_i}$, $\boldsymbol{\xi}_{L\mathbf{v}} := (\xi_1, \dots, \xi_r)$. Note that $\xi_i \neq 1$ for any i . Then we have

$$\sum_{\lambda=1}^{N\eta-1} (\xi_{\mathbf{x}} \boldsymbol{\xi}_{L\mathbf{v}}^{\mathbf{m}})^{\lambda} = \begin{cases} -1 & (\text{ord}_{\eta}((\mathbf{x} + \mathbf{m})^t(L\mathbf{v})) = 0), \\ N\eta - 1 & (\text{ord}_{\eta}((\mathbf{x} + \mathbf{m})^t(L\mathbf{v})) > 0). \end{cases}$$

Here we put $\boldsymbol{\xi}_{L\mathbf{v}}^{\mathbf{m}} := \prod_{i=1}^r \xi_i^{m_i}$. It follows that

$$\begin{aligned} \sum_{\mathbf{x} \in R_a} \sum_{\lambda=1}^{N\eta-1} \xi_{\mathbf{x}}^{\lambda} \zeta_N(s, L\mathbf{v}, \mathbf{x}, \boldsymbol{\xi}_{L\mathbf{v}}^{\lambda}) &= \sum_{\mathbf{x} \in R_a} \sum_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^r} \sum_{\lambda=1}^{N\eta-1} (\xi_{\mathbf{x}} \boldsymbol{\xi}_{L\mathbf{v}}^{\mathbf{m}})^{\lambda} N((\mathbf{x} + \mathbf{m})^t(L\mathbf{v}))^{-s} \\ &= - \sum_{z \in R} Nz^{-s} + N\eta \sum_{z \in \mathbb{Z}, \text{ord}_{\eta} z > 0} Nz^{-s}. \end{aligned} \quad (7)$$

Similarly we obtain

$$\begin{aligned} \sum_{\mathbf{x} \in R_a} \sum_{\lambda=1}^{N\eta-1} \xi_{\mathbf{x}}^{\lambda} \zeta(s, L\mathbf{v}, \mathbf{x}, \boldsymbol{\xi}_{L\mathbf{v}}^{\lambda}) &= - \sum_{z \in R} z^{-s} + N\eta \sum_{z \in \mathbb{Z}, \text{ord}_{\eta} z > 0} z^{-s} \\ &= \zeta(s, R) - N\eta \zeta(s, \{z \in R \mid \text{ord}_{\eta} z > 0\}). \end{aligned} \quad (8)$$

By Lemma 2, we have for $\mathbf{x} \in R_a$

$$\zeta_N(0, L\mathbf{v}, \mathbf{x}, \boldsymbol{\xi}_{L\mathbf{v}}^{\lambda}) = \zeta(0, L\mathbf{v}, \mathbf{x}, \boldsymbol{\xi}_{L\mathbf{v}}^{\lambda}). \quad (9)$$

Then the assertion follows from (6), (7), (8), (9). \square

Dasgupta's p -adic integration $\int d\nu_{\eta}(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, x)$ is originally associated with special values of multiple zeta functions “with the norm” $\zeta_N(\dots)$. By the above Lemma, we can rewrite it in terms of special values of multiple zeta functions “without the norm” $\zeta(\dots)$. This observation is one of the main discoveries in this paper.

Lemma 4. Let $\nu_\eta(\mathfrak{b}, D, \mathcal{U})$, $\mathbf{O} = \mathcal{O}_{F_p} - \pi \mathcal{O}_{F_p}$ be as in Definition 12. Assume that η is good for D . Then we have for $k \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_p$

$$\begin{aligned} \int_{\mathbf{O}} x^k d\nu_\eta(\mathfrak{b}, D, x) &= \zeta(-k, F_{\mathfrak{f}}^\times \cap \mathfrak{b}^{-1} \cap D \cap \mathbf{O}) - N\eta \zeta(-k, F_{\mathfrak{f}}^\times \cap \mathfrak{b}^{-1} \eta \cap D \cap \mathbf{O}), \\ \int_{\mathbf{O}} \langle x \rangle^{-s} d\nu_\eta(\mathfrak{b}, D, x) &= \zeta_p(s, F_{\mathfrak{f}}^\times \cap \mathfrak{b}^{-1} \cap D \cap \mathbf{O}) - N\eta \zeta_p(s, F_{\mathfrak{f}}^\times \cap \mathfrak{b}^{-1} \eta \cap D \cap \mathbf{O}). \end{aligned} \quad (10)$$

Here we put $\langle x \rangle := p^{-\text{ord}_p x} \theta_p(x)^{-1} x$ by using ord_p, θ_p in (2).

Proof. It is enough to show the statement when $D = C(\mathbf{v})$ with $\mathbf{v} = (v_1, \dots, v_r)$, $v_i \in \mathcal{O}_F - \eta$. By definition we can write

$$\int_{\mathbf{O}} x^k d\nu_\eta(\mathfrak{b}, C(\mathbf{v}), x) = \lim_{\leftarrow} \sum_{\bar{a} \in \mathbf{O}/(1+\mathfrak{p}^m \mathcal{O}_{F_p})} a^k \nu_\eta(\mathfrak{b}, C(\mathbf{v}), a(1 + \mathfrak{p}^m \mathcal{O}_{F_p})).$$

By Lemmas 2, 3, we have

$$a^k \nu_\eta(\mathfrak{b}, C(\mathbf{v}), a(1 + \mathfrak{p}^m \mathcal{O}_{F_p})) = - \sum_{\mathbf{m} \in \mathbb{N}^r} \sum_{\lambda=1}^{N\eta-1} \sum_{\mathbf{x} \in R_a} \frac{\xi_{\mathbf{x}}^\lambda \sum_{1 \leq l_i \leq m_i} \left\{ \frac{\mathbf{m}}{\mathbf{l}} \right\} a^k}{(1 - \xi_{L\mathbf{v}}^\lambda)^m},$$

where $L, R_a, \xi_{\mathbf{x}}, \xi_{L\mathbf{v}}$ are as in the proof of Lemma 3. On the other hand, by Lemma 2 again, we obtain

$$\begin{aligned} &\zeta(-k, F_{\mathfrak{f}}^\times \cap \mathfrak{b}^{-1} \cap C(\mathbf{v}) \cap \mathbf{O}) - N\eta \zeta(-k, F_{\mathfrak{f}}^\times \cap \mathfrak{b}^{-1} \eta \cap C(\mathbf{v}) \cap \mathbf{O}) \\ &= - \sum_{\bar{a} \in \mathbf{O}/(1+\mathfrak{p}^m \mathcal{O}_{F_p})} \sum_{\mathbf{x} \in R_a} \sum_{\mathbf{m} \in \mathbb{N}^r} \sum_{\lambda=1}^{N\eta-1} \frac{\xi_{\mathbf{x}}^\lambda \sum_{1 \leq l_i \leq m_i} \left\{ \frac{\mathbf{m}}{\mathbf{l}} \right\} ((\mathbf{x} - \mathbf{l})^t(L\mathbf{v}))^k}{(1 - \xi_{L\mathbf{v}}^\lambda)^m}. \end{aligned}$$

By definition, we see that $L \in \mathfrak{p}^m$, $\mathbf{x}^t(L\mathbf{v}) \equiv a \pmod{\mathfrak{p}^m \mathcal{O}_{F_p}}$ for $\mathbf{x} \in R_a$. It follows that

$$a^k \equiv ((\mathbf{x} - \mathbf{l})^t(L\mathbf{v}))^k \pmod{\mathfrak{p}^m \mathcal{O}_{F_p}} \quad (\mathbf{x} \in R_a),$$

Hence the first assertion is clear. The second assertion follows from the p -adic interpolation property (4). \square

Proof of Theorem 1. For a fractional ideal \mathfrak{b} , a Shintani set D , an open compact subset $\mathcal{U} \subset \mathcal{O}_{F_p}$, and for $* = \emptyset, p$, we put

$$\begin{aligned} L\Gamma_*(\mathfrak{b}, D, \mathcal{U}) &:= L\Gamma_*(F_{\mathfrak{f}}^\times \cap \mathfrak{b}^{-1} \cap D \cap \mathcal{U}), \\ L\Gamma_{\eta,*}(\mathfrak{b}, D, \mathcal{U}) &:= L\Gamma_*(\mathfrak{b}, D, \mathcal{U}) - N\eta L\Gamma_*(\mathfrak{b}\eta^{-1}, D, \mathcal{U}) \end{aligned}$$

whenever each function is well-defined. It suffices to show the following three equalities:

$$\log_p(\epsilon_\eta(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \pi) \pi^{\zeta_{\mathfrak{f}, \eta}(0, (\frac{H_{\mathfrak{f}}/F}{\mathfrak{b}}))}) = [L\Gamma_\eta(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \mathbf{O})]_p, \quad (11)$$

$$\log_p\left(\int_{\mathbf{O}} x d\nu_\eta(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, x)\right) = -L\Gamma_{\eta,p}(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \mathbf{O}), \quad (12)$$

$$L\Gamma_p(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \mathbf{O}) - [L\Gamma(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \mathbf{O})]_p = Y_p((\frac{H/F}{\mathfrak{b}})). \quad (13)$$

Let \mathbf{v}_j, ϵ_j ($j \in J'$) be as in Definition 11-(iv). Since $\mathcal{D}_{\mathfrak{f}}, \pi^{-1}\mathcal{D}_{\mathfrak{f}}$ are fundamental domains of $F \otimes \mathbb{R}_+/E_{\mathfrak{f},+}$, we see that

$$\epsilon \mathcal{D}_{\mathfrak{f}} \cap \pi^{-1}\mathcal{D}_{\mathfrak{f}} = \coprod_{j \in J', \epsilon_j = \epsilon} \epsilon_j C(\mathbf{v}_j) \quad (\epsilon \in E_{\mathfrak{f},+})$$

Namely we have

$$\epsilon_{\eta}(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \pi) = \prod_{j \in J'} \epsilon_j^{\nu_{\eta}(\mathfrak{b}, \epsilon_j C(\mathbf{v}_j), \mathcal{O}_{F_{\mathfrak{p}}})}.$$

By Lemma 3, we can write

$$\begin{aligned} & \nu_{\eta}(\mathfrak{b}, \epsilon_j C(\mathbf{v}_j), \mathcal{O}_{F_{\mathfrak{p}}}) \\ &= \zeta(0, F_{\mathfrak{f}}^{\times} \cap \mathfrak{b}^{-1} \cap \epsilon_j C(\mathbf{v}_j) \cap \mathcal{O}_{F_{\mathfrak{p}}}) - N\eta \zeta(0, F_{\mathfrak{f}}^{\times} \cap \mathfrak{b}^{-1} \eta \cap \epsilon_j C(\mathbf{v}_j) \cap \mathcal{O}_{F_{\mathfrak{p}}}). \end{aligned}$$

Therefore by Proposition 1-(i) we obtain

$$\begin{aligned} \log_p(\epsilon_{\eta}(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \pi)) &= \left[\sum_{j \in J'} L\Gamma_{\eta}(\mathfrak{b}, C(\mathbf{v}_j), \mathcal{O}_{F_{\mathfrak{p}}}) - \sum_{j \in J'} L\Gamma_{\eta}(\mathfrak{b}, \epsilon_j C(\mathbf{v}_j), \mathcal{O}_{F_{\mathfrak{p}}}) \right]_p \\ &= [L\Gamma_{\eta}(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \mathcal{O}_{F_{\mathfrak{p}}}) - L\Gamma_{\eta}(\mathfrak{b}, \pi^{-1}\mathcal{D}_{\mathfrak{f}}, \mathcal{O}_{F_{\mathfrak{p}}})]_p. \end{aligned} \quad (14)$$

We easily see that

$$\begin{aligned} \zeta_{\mathfrak{f}, \eta}(0, (\frac{H_{\mathfrak{f}}/F}{\mathfrak{b}})) &= \nu_{\eta}(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \mathcal{O}_{F_{\mathfrak{p}}}), \\ \pi(F_{\mathfrak{f}}^{\times} \cap \mathfrak{b}^{-1} \cap \pi^{-1}\mathcal{D}_{\mathfrak{f}} \cap \mathcal{O}_{F_{\mathfrak{p}}}) &= F_{\mathfrak{f}}^{\times} \cap \pi \mathfrak{b}^{-1} \cap \mathcal{D}_{\mathfrak{f}} \cap \pi \mathcal{O}_{F_{\mathfrak{p}}}. \end{aligned}$$

Hence, by Proposition 1-(i) again, we get

$$\log_p(\pi^{\zeta_{\mathfrak{f}, \eta}(0, (\frac{H_{\mathfrak{f}}/F}{\mathfrak{b}}))}) = [L\Gamma_{\eta}(\mathfrak{b}, \pi^{-1}\mathcal{D}_{\mathfrak{f}}, \mathcal{O}_{F_{\mathfrak{p}}}) - L\Gamma_{\eta}(\pi^{-1}\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \pi \mathcal{O}_{F_{\mathfrak{p}}})]_p. \quad (15)$$

Since $(F_{\mathfrak{f}}^{\times} \cap \mathfrak{b}^{-1} \cap \mathcal{D}_{\mathfrak{f}} \cap \mathbf{O}) \coprod (F_{\mathfrak{f}}^{\times} \cap \pi \mathfrak{b}^{-1} \cap \mathcal{D}_{\mathfrak{f}} \cap \pi \mathcal{O}_{F_{\mathfrak{p}}}) = F_{\mathfrak{f}}^{\times} \cap \mathfrak{b}^{-1} \cap \mathcal{D}_{\mathfrak{f}} \cap \mathcal{O}_{F_{\mathfrak{p}}}$, we have

$$L\Gamma_{\eta}(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \mathbf{O}) = L\Gamma_{\eta}(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \mathcal{O}_{F_{\mathfrak{p}}}) - L\Gamma_{\eta}(\pi^{-1}\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \pi \mathcal{O}_{F_{\mathfrak{p}}}). \quad (16)$$

Then the assertion (11) follows from (14), (15), (16).

Next, differentiating (10) at $s = 0$, we obtain

$$- \int_{\mathbf{O}} \log_p x d\nu_{\eta}(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, x) = L\Gamma_{\eta, p}(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \mathbf{O}).$$

By definition, we have $\log_p(\oint_{\mathbf{O}} x d\nu_{\eta}(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, x)) = \int_{\mathbf{O}} \log_p x d\nu_{\eta}(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, x)$. Hence the assertion (12) is clear.

Finally we prove (13). Let D be a Shintani domain mod E_+ . For each $c \in C_{\mathfrak{fp}}$, we take an integral ideal \mathfrak{a}_c satisfying $\mathfrak{a}_c \mathfrak{f} \in \pi(c)$, and put $R_c := \{z \in D \mid \mathcal{O}_F \supset z \mathfrak{a}_c \mathfrak{fp} \in c\}$. By (5) we can write

$$Y_p((\frac{H/F}{\mathfrak{b}})) = \sum_{c \in C_{\mathfrak{fp}}, \text{Art}(\bar{c})|_H = (\frac{H/F}{\mathfrak{b}})} L\Gamma_p(R_c) - \left[\sum_{c \in C_{\mathfrak{fp}}, \text{Art}(\bar{c})|_H = (\frac{H/F}{\mathfrak{b}})} L\Gamma(R_c) \right]_p.$$

Since H is the fixed subfield under $(\frac{H_{\mathfrak{f}}/F}{\mathfrak{p}})$, we may replace

$$\sum_{c \in C_{\mathfrak{f}\mathfrak{p}}, \text{Art}(\bar{c})|_H = (\frac{H/F}{\mathfrak{b}})} \cdots = \sum_{k=0}^{e-1} \sum_{c \in C_{\mathfrak{f}\mathfrak{p}}, \bar{c} = [\mathfrak{b}\mathfrak{p}^{-k}]} \cdots,$$

where \bar{c} denotes the image under $C_{\mathfrak{f}\mathfrak{p}} \rightarrow C_{\mathfrak{f}}$, $[\mathfrak{a}]$ denotes the ideal class in $C_{\mathfrak{f}}$ of a fractional ideal \mathfrak{a} . On the other hand, we can write for $* = \emptyset, p$

$$L\Gamma_*(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \mathbf{O}) = \sum_{k=0}^{e-1} L\Gamma_*(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \mathfrak{p}^k \mathcal{O}_{F_{\mathfrak{p}}}^{\times}).$$

Therefore it suffices to show that we have for each k

$$\begin{aligned} & \left(\sum_{c \in C_{\mathfrak{f}\mathfrak{p}}, \bar{c} = [\mathfrak{b}\mathfrak{p}^{-k}]} L\Gamma_p(R_c) \right) - L\Gamma_p(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \mathfrak{p}^k \mathcal{O}_{F_{\mathfrak{p}}}^{\times}) \\ &= \left[\left(\sum_{c \in C_{\mathfrak{f}\mathfrak{p}}, \bar{c} = [\mathfrak{b}\mathfrak{p}^{-k}]} L\Gamma(R_c) \right) - L\Gamma(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \mathfrak{p}^k \mathcal{O}_{F_{\mathfrak{p}}}^{\times}) \right]_p. \end{aligned} \quad (17)$$

We fix k . Whenever $\bar{c} = [\mathfrak{b}\mathfrak{p}^{-k}]$, $\pi(c) \in C_{(1)}$ is constant, so we may put \mathfrak{a}_c to be a fixed integral ideal \mathfrak{a}_0 . Then we have

$$\coprod_{c \in C_{\mathfrak{f}\mathfrak{p}}, \bar{c} = [\mathfrak{b}\mathfrak{p}^{-k}]} R_c = \{z \in (\mathfrak{a}_0 \mathfrak{f}\mathfrak{p})^{-1} \cap D \mid (z\mathfrak{a}_0 \mathfrak{f}\mathfrak{p}, \mathfrak{f}\mathfrak{p}) = 1, [z\mathfrak{a}_0 \mathfrak{f}\mathfrak{p}] = [\mathfrak{b}\mathfrak{p}^{-k}] \text{ in } C_{\mathfrak{f}}\}.$$

Let $\alpha_0 \in F_+$ be a generator of the principal ideal $(\mathfrak{a}_0 \mathfrak{f}\mathfrak{p})(\mathfrak{b}\mathfrak{p}^{-k})^{-1}$. Then the following are equivalent:

$$[z\mathfrak{a}_0 \mathfrak{f}\mathfrak{p}] = [\mathfrak{b}\mathfrak{p}^{-k}] \Leftrightarrow [(z\alpha_0)] = [(1)] \Leftrightarrow \exists \epsilon \in E_+ \text{ s.t. } z\epsilon\alpha_0 \equiv 1 \pmod{\mathfrak{f}}.$$

Hence, taking a representative set E_0 of $E_+/E_{\mathfrak{f},+}$, we can write

$$\begin{aligned} & \{z \in (\mathfrak{a}_0 \mathfrak{f}\mathfrak{p})^{-1} \cap D \mid (z\mathfrak{a}_0 \mathfrak{f}\mathfrak{p}, \mathfrak{f}\mathfrak{p}) = 1, [z\mathfrak{a}_0 \mathfrak{f}\mathfrak{p}] = [\mathfrak{b}\mathfrak{p}^{-k}] \text{ in } C_{\mathfrak{f}}\} \\ &= \coprod_{\epsilon \in E_0} (\epsilon\alpha_0)^{-1} (F_{\mathfrak{f}}^{\times} \cap \mathfrak{b}^{-1} \cap \epsilon\alpha_0 D \cap \mathfrak{p}^k \mathcal{O}_{F_{\mathfrak{p}}}^{\times}). \end{aligned}$$

Namely we have for $* = \emptyset, p$

$$\sum_{c \in C_{\mathfrak{f}\mathfrak{p}}, \bar{c} = [\mathfrak{b}\mathfrak{p}^{-k}]} L\Gamma_*(R_c) = \sum_{\epsilon \in E_0} L\Gamma_*((\epsilon\alpha_0)^{-1} (F_{\mathfrak{f}}^{\times} \cap \mathfrak{b}^{-1} \cap \epsilon\alpha_0 D \cap \mathfrak{p}^k \mathcal{O}_{F_{\mathfrak{p}}}^{\times})). \quad (18)$$

On the other hand, $\mathcal{D}'_{\mathfrak{f}} := \coprod_{\epsilon \in E_0} \epsilon\alpha_0 D$ becomes another Shintani domain mod $E_{\mathfrak{f},+}$, and we can write for $* = \emptyset, p$

$$L\Gamma_*(\mathfrak{b}, \mathcal{D}'_{\mathfrak{f}}, \mathfrak{p}^k \mathcal{O}_{F_{\mathfrak{p}}}^{\times}) = \sum_{\epsilon \in E_0} L\Gamma_*(F_{\mathfrak{f}}^{\times} \cap \mathfrak{b}^{-1} \cap \epsilon\alpha_0 D \cap \mathfrak{p}^k \mathcal{O}_{F_{\mathfrak{p}}}^{\times}). \quad (19)$$

Then the assertion (17), replacing $\mathcal{D}_{\mathfrak{f}}$ with $\mathcal{D}'_{\mathfrak{f}}$, follows from (18), (19) and Proposition 1. We conclude the proof of (13) by showing that

$$L\Gamma_p(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \mathfrak{p}^k \mathcal{O}_{F_{\mathfrak{p}}}^{\times}) - L\Gamma_p(\mathfrak{b}, \mathcal{D}'_{\mathfrak{f}}, \mathfrak{p}^k \mathcal{O}_{F_{\mathfrak{p}}}^{\times}) = [L\Gamma(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \mathfrak{p}^k \mathcal{O}_{F_{\mathfrak{p}}}^{\times}) - L\Gamma(\mathfrak{b}, \mathcal{D}'_{\mathfrak{f}}, \mathfrak{p}^k \mathcal{O}_{F_{\mathfrak{p}}}^{\times})]_p.$$

Note that the independence on the choice of $\mathcal{D}_{\mathfrak{f}}$ is also discussed in [Da, §5.2] under certain conditions. Similarly to [Yo, Chap. III, Lemma 3.13], we see that there exist cones $C(\mathbf{v}_j)$ and units $u_j \in E_{\mathfrak{f},+}$ ($j \in J''$) which satisfy

$$\mathcal{D}_{\mathfrak{f}} = \coprod_{j \in J''} C(\mathbf{v}_j), \quad \mathcal{D}'_{\mathfrak{f}} = \coprod_{j \in J''} u_j C(\mathbf{v}_j).$$

Therefore it suffices to show that

$$\begin{aligned} & L\Gamma_p(\mathfrak{b}, C(\mathbf{v}_j), \mathfrak{p}^k \mathcal{O}_{F_{\mathfrak{p}}}^{\times}) - L\Gamma_p(\mathfrak{b}, u_j C(\mathbf{v}_j), \mathfrak{p}^k \mathcal{O}_{F_{\mathfrak{p}}}^{\times}) \\ &= [L\Gamma(\mathfrak{b}, C(\mathbf{v}_j), \mathfrak{p}^k \mathcal{O}_{F_{\mathfrak{p}}}^{\times}) - L\Gamma(\mathfrak{b}, u_j C(\mathbf{v}_j), \mathfrak{p}^k \mathcal{O}_{F_{\mathfrak{p}}}^{\times})]_p. \end{aligned}$$

It follows from Proposition 1 since $F_{\mathfrak{f}}^{\times} \cap \mathfrak{b}^{-1} \cap u_j C(\mathbf{v}_j) \cap \mathfrak{p}^k \mathcal{O}_{F_{\mathfrak{p}}}^{\times} = u_j (F_{\mathfrak{f}}^{\times} \cap \mathfrak{b}^{-1} \cap C(\mathbf{v}_j) \cap \mathfrak{p}^k \mathcal{O}_{F_{\mathfrak{p}}}^{\times})$. \square

References

- [CN1] P. Cassou-Noguès, Analogues p -adiques de quelques fonctions arithmétiques, *Publ. Math. Bordeaux* (1974-1975), 1–43.
- [CN2] P. Cassou-Noguès, Valeurs aux entiers négative des fonction zêta et fonction zêta p -adiques, *Inv. Math.* **51** (1979), 29–59.
- [Da] S. Dasgupta, Shintani zeta-functions and Gross-Stark units for totally real fields, *Duke Math. J.* **143** (2008), no. 2, 225–279.
- [DDP] S. Dasgupta, H. Darmon, R. Pollack, Hilbert modular forms and the Gross-Stark conjecture, *Ann. of Math. (2)* **174** (2011), no. 1, 439–484 .
- [Gr] B. H. Gross, p -adic L -series at $s = 0$, *J. Fac. Sci. Univ. Tokyo* **28** (1981), 979–994.
- [Ka1] T. Kashio, On a p -adic analogue of Shintani’s formula, *J. Math. Kyoto Univ.* **45** (2005), 99–128.
- [Ka2] T. Kashio, Fermat curves and a refinement of the reciprocity law on cyclotomic units, *J. Reine Angew. Math.*, DOI: 10.1515/crelle-2015-0081.
- [Ka3] T. Kashio, On the ratios of Barnes’ multiple gamma functions to the p -adic analogues (arXiv:1703.10411),
- [KY1] T. Kashio, and H. Yoshida, On p -adic absolute CM-Periods, I, *Amer. J. Math.* **130** (2008), no. 6, 1629–1685.

- [KY2] T. Kashio, and H. Yoshida, On p -adic absolute CM-Periods, II, *Publ. Res. Inst. Math. Sci.* **45** (2009), no. 1, 187–225.
- [Sh] T. Shintani, On evaluation of zeta functions of totally real algebraic number fields at non-positive integers, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **23** (1976), no. 2, 393–417.
- [Yo] H. Yoshida, *Absolute CM-Periods*, Math. Surveys Monogr. **106**, Amer. Math. Soc., Providence, RI, 2003.